PARABOLIC GEOMETRIES FOR PEOPLE THAT LIKE PICTURES

LECTURE 3: GEOMETRY AS "LIE THEORY WITH EXTRA STEPS" (PART 2)

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Last time, we saw that Euclidean geometry could be reformulated in terms of the pair $(\mathrm{I}(2), \mathrm{O}(2))$, defining things like circles and lines in terms of one-parameter subgroups and cosets. This time, we will look at other homogeneous geometries determined by pairs (G, H). Specifically, we will be looking at three types of geometries:

- Spherical geometry $(\mathrm{O}(1+2), \mathrm{O}(2))$
- Hyperbolic geometry $(\mathrm{PO}(1,2), \mathrm{O}(2))$
- Affine geometry $(\mathrm{Aff}(2), \mathrm{GL}_2 \mathbb{R})$

This way of thinking, where symmetries are used to determine and categorize the structures of various geometries, is often referred to as the *Erlangen program*, introduced by Felix Klein toward the end of the nineteenth century.

By the end of this lecture, we should see that the intuition we developed for Euclidean geometry extends, with some small adjustments, fairly easily to these other geometries. In the next lecture, we will use the 1-dimensional model geometry $(\mathrm{SL}_2 \mathbb{R}, B)$ to introduce some basic aspects of parabolic geometries.

1. Model geometries

To begin, we define the notion of geometry on which the rest of the course will be based.

Definition 1.1. A *model geometry*[1] (or simply *model*) is a pair (G, H), where G is a Lie group and $H \leq G$ is a closed subgroup such that G/H is connected. In a model (G, H), the Lie group G is called the *model group* and H is called the *isotropy* or *stabilizer subgroup*.

For example, the geometry of the Euclidean plane came from the model $(\mathrm{I}(2), \mathrm{O}(2))$; $\mathrm{O}(2)$ is a closed subgroup of $\mathrm{I}(2)$, since it is the stabilizer of 0 in the connected manifold $\mathbb{R}^2 \cong \mathrm{I}(2)/\mathrm{O}(2)$.

[1] We sometimes also call these *Klein geometries*, though we usually use this term to refer specifically to the geometric structure of the model geometry expressed in terms of a Cartan connection.

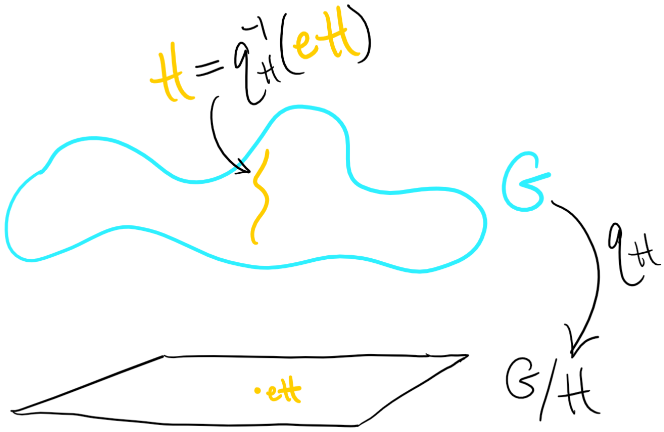

FIGURE 1. In a model (G, H), the model group G can be thought of as the bundle of observer configurations over the space G/H, and the isotropy H runs through all the configurations lying over a given point in G/H

While the jump in abstraction might seem intimidating at first, there is really not much more going on here than there is in Euclidean geometry. For a given model (G, H), we are describing a geometric structure on the manifold G/H. We can, as we did before, think of the model group G as the bundle of configurations for ourselves as **observers** wandering the geometry on G/H, with bundle map given by the natural quotient map

$$q_H : G \to G/H, g \mapsto gH.$$

The isotropy H, then, describes the space of configurations that can occur over a point of G/H; for each $g \in G$, we can reach every other configuration lying over $q_H(g)$ by right-translating by an element of H. In particular, since H acts freely and transitively on each fiber of G over G/H by right-translation, G is a *principal H-bundle*[2] over G/H.

On top of giving us a way to place ourselves inside the geometry as observers, the model group G naturally acts on both itself and G/H from the left, and this action defines what symmetry means for the model geometry. In Euclidean geometry, for example, the model group is precisely the group of transformations preserving the Euclidean structure: the isometry group. In a model geometry (G, H), elements of G play the same role that isometries do in Euclidean geometry, acting as transformations that preserve the underlying geometric structure.

What *is* the geometric structure of (G, H)? This is the wonderfully elegant idea of Klein's Erlangen program: **the geometric structure is whatever is preserved by the symmetries of the geometry**!

[2]Recall that a principal H-bundle $\mathscr{G}$ over M is a fiber bundle over M together with a right-action of H on $\mathscr{G}$ that is free and transitive on each fiber.

Since Euclidean isometries send lines to lines and circles to circles, the notions of line and circle make sense inside of Euclidean geometry. Similarly, geometric objects of a model geometry (G, H) are going to be things preserved by the (left-)action of G.

"Definition" 1.2. We will say that something is *geometric* for the model (G, H) if and only if it is preserved by some action of G induced by the natural left-action of G on itself.

Of course, one thing that is *always* preserved by the model group G acting on itself from the left is its Maurer-Cartan form ω_G, since it is left-invariant by definition! Indeed, if we want to think of the geometric structure of (G, H) as an explicit diffeo-geometric object, then that diffeo-geometric object *is* the Maurer-Cartan form ω_G on the principal H-bundle G over G/H, since the symmetries of the Maurer-Cartan form are precisely the left-translations by elements of G. To convey this equivalence rigorously, we include the following optional proposition, which is more or less the same as Corollary 3.4.11 in [1].

Proposition 1.3. *If $f : G \to G$ is a map such that $f^*\omega_G = \omega_G$ and $f(gh) = f(g)h$ for all $g \in G$ and $h \in H$, then there is some $a \in G$ such that $f = \mathrm{L}_a$.*

Proof. Denote by $\mu : G \times G \to G$ the group operation $(g, g') \mapsto gg'$ and by $(\cdot)^{-1} : G \to G$ the inverse operation $g \mapsto g^{-1}$.

Let $\sigma : G \to G$ be given by

$$g \mapsto f(g)g^{-1} = (\mu \circ (f, (\cdot)^{-1}))(g, g).$$

We want to show that σ is constant, because then

$$\sigma(e)g = \sigma(g)g = (f(g)g^{-1})g = f(g)$$

for all $g \in G$, so that $f = \mathrm{L}_{\sigma(e)}$. Since $f(gh) = f(g)h$,

$$\sigma(gh) = f(gh)h^{-1}g^{-1} = f(g)g^{-1} = \sigma(g)$$

for all $g \in G$ and $h \in G$, so σ is invariant under right-translation by H. In particular, since G/H is connected, σ is constant if and only if it is constant on a connected component of G, hence it suffices to prove that $\sigma^*\omega_G = 0$.

For $X \in T_gG$ and $Y \in T_{g'}G$,

$$\mu^*\omega_G(X, Y) = \mathrm{Ad}_{(g')^{-1}}(\omega_G(X)) + \omega_G(Y)$$

and

$$(\cdot)^{-1*}\omega_G(X) = -\mathrm{Ad}_g(\omega_G(X)).$$

Thus,

$$\begin{aligned}\sigma^*\omega_{G}(X) &= (f,(\cdot)^{-1})^*(\mu^*\omega_{G})(X,X) = \mu^*\omega_{G}(f_*X,(\cdot)^{-1}_*X)\\ &= \mathrm{Ad}_{((\cdot)^{-1}(g))^{-1}}(\omega_{G}(f_*X)) + \omega_{G}((\cdot)^{-1}_*X)\\ &= \mathrm{Ad}_g(f^*\omega_{G}(X)) + (\cdot)^{-1*}\omega_{G}(X)\\ &= \mathrm{Ad}_g(\omega_{G}(X)) - \mathrm{Ad}_g(\omega_{G}(X)) = 0. \quad \square\end{aligned}$$

2. Spherical geometry

Last time, we explained how circles and angles in Euclidean geometry ultimately came from the subgroup $\mathrm{O}(2) < \mathrm{I}(2)$, which was the stabilizer of the point $0 \in \mathbb{R}^2 \cong \mathrm{I}(2)/\mathrm{O}(2)$. Using our new terminology, this was clearly referring to $\mathrm{O}(2)$ as the isotropy subgroup of the model $(\mathrm{I}(2), \mathrm{O}(2))$. Now, we will investigate what another geometry with isotropy $\mathrm{O}(2)$ looks like.

Consider an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ with the usual Euclidean structure from the dot product. The Lie group $\mathrm{O}(3)$ acts on $\mathbb{R}^3$ by linear isometries (by definition), and we get a copy of $\mathrm{O}(2)$ in $\mathrm{O}(3)$ as the subgroup stabilizing the vector e_1. This points us toward a new model geometry: $(\mathrm{O}(3), \mathrm{O}(2))$, also called (2-dimensional) spherical geometry.

Since $\mathrm{O}(2)$ is the stabilizer of e_1, we can identify $\mathrm{O}(3)/\mathrm{O}(2)$ with $\mathrm{O}(3) \cdot e_1 = S^2$, the unit 2-sphere in $\mathbb{R}^3$. As before with $I(2)$, we can think of $\mathrm{O}(3)$ as the orthonormal frame bundle over S^2, with bundle map $q_{\mathrm{O}(2)} : \mathrm{O}(3) \to S^2 \cong \mathrm{O}(3)/\mathrm{O}(2)$ given by $g \mapsto g \cdot e_1$.

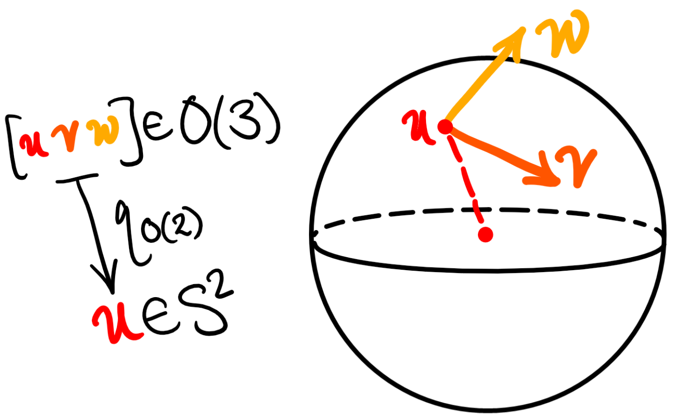

Figure 2. The Lie group $\mathrm{O}(3)$ thought of as the orthonormal frame bundle of $S^2 \cong \mathrm{O}(3)/\mathrm{O}(2)$

At the identity element $\mathbb{1} = [e_1\, e_2\, e_3] \in \mathrm{O}(3)$, we have the tangent space $T_{\mathbb{1}}\,\mathrm{O}(3)$, which we identify with the Lie algebra

$$\mathfrak{o}(3) = \left\{ \begin{bmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Considering $\mathfrak{i}(2)$ and $\mathfrak{o}(3)$ as O(2)-representations, using the restriction of their adjoint representations to their copy of O(2), we have an isomorphism of O(2)-representations

$$\rho_+ : \mathfrak{i}(2) \to \mathfrak{o}(3)$$

given by

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix}\right) \mapsto \begin{bmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{bmatrix}.$$

In particular, O(2) behaves the same way on the subspace $\rho_+(\mathbb{R}^2)$ as it does on the subalgebra of translations $\mathbb{R}^2 < \mathfrak{i}(2)$. The subspace $\rho_+(\mathbb{R}^2)$ is not itself a subalgebra, though the one-parameter subgroups it generates can be thought of as "translations" in spherical geometry.

Writing $\{\bar{e}_1, \bar{e}_2\}$ for the usual orthonormal basis for $\mathbb{R}^2$, with bars to distinguish them from e_1 and e_2 in $\mathbb{R}^3$, we can consider the one-parameter subgroup $\exp(t\rho_+(\bar{e}_1))$ corresponding to

$$\rho_+(\bar{e}_1) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

When acting on the left, $\exp(t\rho_+(\bar{e}_1))$ behaves as a transformation, rotating the sphere in a way that preserves the "equator" given by the intersection of the sphere with the plane $\langle e_1, e_2 \rangle$ generated by e_1 and e_2 in $\mathbb{R}^3$.

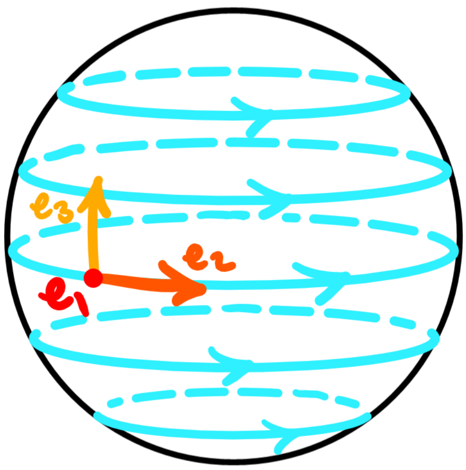

FIGURE 3. Acting on the left by the one-parameter subgroup of transformations $\exp\left(t \left[\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right)$ rotates the sphere in a way that preserves its intersection with the plane $\langle e_1, e_2 \rangle$ generated by e_1 and e_2

In general, the "translations" $\exp(t\rho_+(x\bar{e}_1 + y\bar{e}_2))$ will preserve the great circle given by the intersection of the sphere with $\langle e_1, xe_2 + ye_3 \rangle$.

Definition 2.1. A *great circle* in spherical geometry is a subset given by the intersection of the sphere with a 2-dimensional subspace of $\mathbb{R}^3$ (as a vector space).

Because $\mathrm{O}(3)$ acts linearly on $\mathbb{R}^3$, it sends 2-dimensional subspaces to 2-dimensional subspaces, so since $\mathrm{O}(3)$ also preserves the sphere, it sends great circles to great circles. In other words, the notion of great circle is preserved under the action of the model group $\mathrm{O}(3)$, hence great circles are geometric objects in spherical geometry. We can think of great circles as the spherical analogue of lines.

Note that only one orbit of $\exp(t\rho_+(\bar{e}_1))$ on S^2 was a great circle; all the others were intersections of the sphere with translations of $\langle e_1, e_2\rangle$ by some multiple of e_3, which are not subspaces of $\mathbb{R}^3$ as a vector space. Such intersections are also preserved by the action of $\mathrm{O}(3)$, but they are not the spherical analogue of lines.

When acting on the right, $\exp(t\rho_+(\bar{e}_1))$ behaves as a motion, which we can think of as the spherical analogue of "walking forward" in Euclidean geometry: for $g \in \mathrm{O}(3)$, $g\exp(t\rho_+(\bar{e}_1))$ is given by starting at g and moving for time t along the great circle $S^2 \cap g\cdot\langle e_1, e_2\rangle$ with velocity $q_{\mathrm{O}(2)*}((\omega_{\mathrm{O}(3)})^{-1}(\rho_+(\bar{e}_1)))$, the tangent vector corresponding to $g \cdot e_2$.

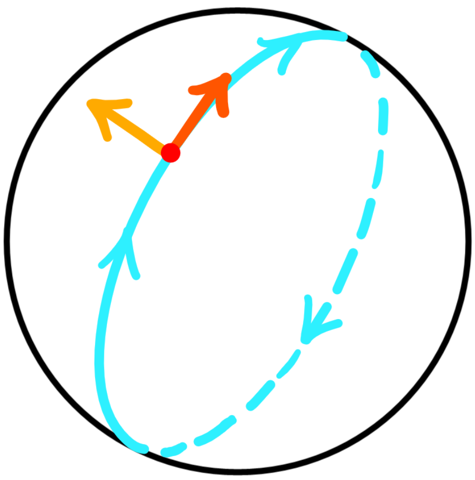

FIGURE 4. Acting on $g \in \mathrm{O}(3)$ from the right by the one-parameter subgroup of motions $\exp(t\rho_+(\bar{e}_1))$ moves along the great circle given by the intersection of the sphere with the plane $g \cdot \langle e_1, e_2\rangle$

In particular, when moving by "translations" in spherical geometry, we always trace out great circles on the sphere, so we could have defined great circles to be subsets of the form $q_{\mathrm{O}(2)}(g\exp(\mathbb{R}\rho_+(v)))$ for some $g \in \mathrm{O}(3)$ and some nonzero $v \in \mathbb{R}^2$, as we did with lines in Euclidean geometry.

Exercise. Using what "translations" look like in spherical geometry and the warm-up discussion on visualizing the Lie bracket, verify that $\rho_+(\mathbb{R}^2)$ is not a subalgebra of $\mathfrak{o}(3)$ *without* doing any computations.

For circles, we can use the same definition as before in Euclidean geometry: pick a center $x \in S^2$, a radius given by some $a \in \mathrm{O}(3)$, and define $C_a(x) := \{q_{\mathrm{O}(2)}(ga) : g \in q_{\mathrm{O}(2)}^{-1}(x)\}$. Again, such sets are given by "all the points in S^2 that some orthonormal frame over x thinks is a away from itself". They happen to be precisely those (nonempty) intersections of the sphere with affine planes from before; in particular, great circles also happen to be special examples of circles in spherical geometry.

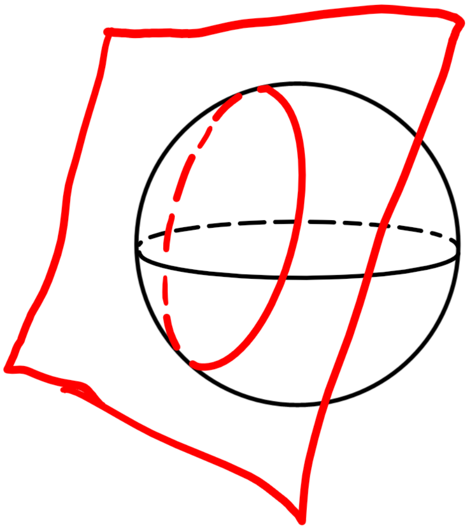

FIGURE 5. The intersection of an affine plane in $\mathbb{R}^3$ with S^2 gives a circle

3. Hyperbolic geometry

Now, consider $\mathbb{R}^3$ with the Minkowski quadratic form Q given by

$$Q(ae_1 + be_2 + ce_3) := a^2 - b^2 - c^2,$$

for which the linear isometries are given by $\mathrm{O}(1,2)$. When we take $\mathrm{O}(1,2)$ and quotient by the center, generated by $-\mathbb{1}$, we get $\mathrm{PO}(1,2)$, which naturally acts on $\mathbb{RP}^2$, the space of 1-dimensional linear subspaces of $\mathbb{R}^3$.

In $\mathbb{R}^3$, the set $Q^{-1}(1)$ is a two-sheeted hyperboloid on which $\mathrm{O}(1,2)$ acts transitively. The two sheets are images of each other under the linear transformation $-\mathbb{1}$, so the image of $Q^{-1}(1)$ in $\mathbb{RP}^2$ is connected and $\mathrm{PO}(1,2)$ acts transitively on it. The stabilizer of the line $\langle e_1\rangle$ is a copy of $\mathrm{O}(2)$, which leads us to consider the model geometry $(\mathrm{PO}(1,2), \mathrm{O}(2))$, also called (2-dimensional) hyperbolic geometry.

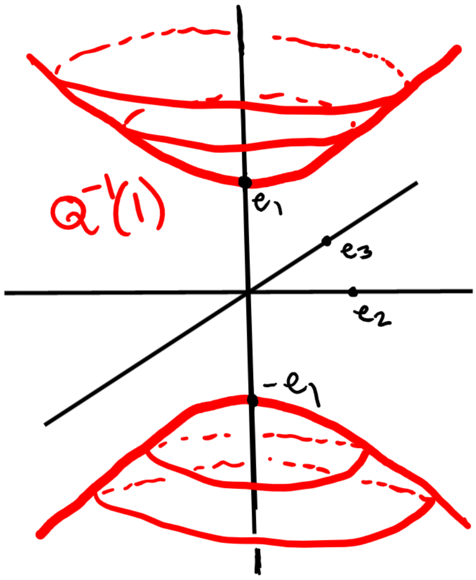

FIGURE 6. A drawing of the two-sheeted hyperboloid $Q^{-1}(1)$ in $\mathbb{R}^3$

Again, we identify $\mathrm{PO}(1,2)$ with the orthonormal frame bundle of $\mathbb{H}^2 \cong \mathrm{PO}(1,2)/\,\mathrm{O}(2)$, with bundle map $q_{\mathrm{O}(2)} : \mathrm{PO}(1,2) \to \mathbb{H}^2$ given by $g \mapsto g \cdot \langle e_1 \rangle$. Indeed, we can topologically identify $\mathbb{H}^2$ with a more familiar space: choosing a sheet of $Q^{-1}(1)$, each point of the sheet projects to a unique point of the plane $\langle e_2, e_3 \rangle$, so we can topologically identify $\mathbb{H}^2$ with $\mathbb{R}^2$.

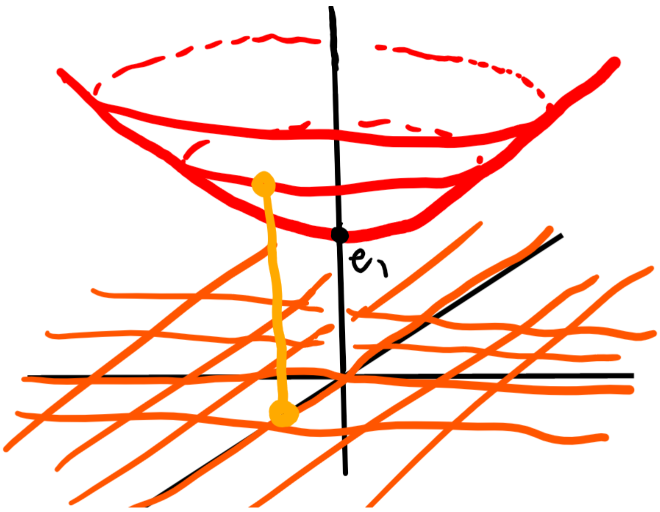

FIGURE 7. Each point of a sheet of $Q^{-1}(1)$ projects to a unique point of the plane $\langle e_2, e_3 \rangle$

As with spherical geometry, there is a convenient isomorphism of O(2)-representations

$$\rho_- : \mathfrak{i}(2) \to \mathfrak{po}(1,2)$$

given by

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix} \right) \mapsto \begin{pmatrix} 0 & x & y \\ x & 0 & -z \\ y & z & 0 \end{pmatrix},$$

so that O(2) behaves the same way on the subspace $\rho_-(\mathbb{R}^2)$ as it does on the subalgebra of translations in Euclidean geometry.

Similar to the above, we get hyperbolic analogues of lines—called *geodesics*—by taking images of (nonempty) intersections with $Q^{-1}(1)$ of 2-dimensional linear subspaces in $\mathbb{R}^3$. Naturally, these geodesics happen to be equivalent to subsets of the form $q_{\mathrm{O}(2)}(g \exp(\mathbb{R}\rho_-(v)))$ for some $g \in \mathrm{PO}(1,2)$ and some nonzero $v \in \mathbb{R}^2$. Circles follow a similar pattern to before as well.

Shamefully, we do not have nearly enough time to give an adequate treatment of the many intricacies of hyperbolic geometry, even in dimension two. Indeed, in order to get to the material that we will actually need later, I might have to omit this whole section from the actual lecture.

I'll sneak some hyperbolic geometry back into the course when we talk about the intuition for the Killing form, but until then, know that hyperbolic geometry is important and I should probably be tried at The Hague for not spending more time on it here.

4. Affine geometry

At the end of the last lecture, we described how the behavior of lines and parallelism in Euclidean geometry came from the closed normal subgroup of translations $\mathbb{R}^2$ acting simply transitively on the Euclidean plane. In other words, in a model geometry (G, H) such that the model group G has a closed normal subgroup isomorphic to $\mathbb{R}^2$ that acts simply transitively on G/H, we should get the same notions of lines and parallelism.

To give an example of this, consider the Lie group Aff(2) of transformations of the plane generated by translations and (not necessarily isometric) linear transformations. This is the group of *affine transformations* of the plane, and by essentially the same argument we made for I(2), we get an isomorphism $\mathrm{Aff}(2) \simeq \mathbb{R}^2 \rtimes \mathrm{GL}_2\,\mathbb{R}$. The model $(\mathrm{Aff}(2), \mathrm{GL}_2\,\mathbb{R})$ gives (2-dimensional) affine geometry.

Instead of the *orthonormal* frame bundle of $\mathbb{R}^2$, we identify Aff(2) with the full frame bundle of $\mathbb{R}^2$.

Definition 4.1. A *frame* over $x \in \mathbb{R}^2$ is a linear isomorphism from $\mathbb{R}^2 \approx T_0\mathbb{R}^2$ to $T_x\mathbb{R}^2$.

As before, elements $g \in \mathrm{Aff}(2)$ are identified with their pushforwards at the origin $g_* : T_0\mathbb{R}^2 \to T_{g(0)}\mathbb{R}^2$, and we get the natural bundle map $q_{\mathrm{GL}_2\mathbb{R}} : \mathrm{Aff}(2) \to \mathbb{R}^2 \cong \mathrm{Aff}(2)/\mathrm{GL}_2\mathbb{R}$ given by $g \mapsto g(0) \cong g\,\mathrm{GL}_2\mathbb{R}$.

It is worth spending some time thinking about what it is like to be a pedestrian on the affine plane. We are accustomed to only being able to rotate on the spot, but in affine geometry, we have a much wider range of options. For example, imagine "rotating" by the unipotent transformations $\left[\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right]$: our notion of "forward" remains the same, but our notion of "left" skews forward by t.

FIGURE 8. Right-translating by $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ looks quite different from just rotating on the spot

We can also rescale ourselves by right-translating by a linear transformation of the form $\lambda\mathbb{1}$ for some $\lambda > 0$. For $\lambda \in (0, 1)$, such transformations shrink us, and for $\lambda \in (1, +\infty)$, they expand us.

FIGURE 9. Right-translating by $\frac{1}{2}\mathbb{1}$ rescales us by $1/2$

We can, again, define lines as subsets of $\mathrm{Aff}(2)/\mathrm{GL}_2\mathbb{R} \cong \mathbb{R}^2$ of the form $q_{\mathrm{GL}_2\mathbb{R}}(g\exp(\mathbb{R}v))$ for some $g \in \mathrm{Aff}(2)$ and some nonzero translational velocity $v \in \mathbb{R}^2 < \mathfrak{aff}(2)$. This definition coincides with the usual notion of line, and parallelism works the same as it does in Euclidean geometry by the argument from last lecture.

Of course, we already knew that lines and parallelism would be the same as before; that was the point. We start to see changes when we try to use the definition of circles from last time. Indeed, for $a \in \mathrm{Aff}(2)$ and $x \in \mathbb{R}^2$, consider the set

$$C_a(x) := \{q_{\mathrm{GL}_2\mathbb{R}}(ga) : g \in q^{-1}_{\mathrm{GL}_2\mathbb{R}}(x)\}.$$

For $a \in \mathrm{GL}_2\mathbb{R}$, $C_a(x)$ is just the point x. For $a \in \tau_v \mathrm{GL}_2\mathbb{R}$ for some nonzero $v \in \mathbb{R}^2$, however,

$$q_{\mathrm{GL}_2\mathbb{R}}(ga) = q_{\mathrm{GL}_2\mathbb{R}}(\tau_x \circ A \circ \tau_v) = q_{\mathrm{GL}_2\mathbb{R}}(\tau_{x+A(v)})$$

for some $A \in \mathrm{GL}_2\mathbb{R}$ such that $g = \tau_x \circ A$, hence $C_a(x)$ is the set $x + \mathrm{GL}_2\mathbb{R} \cdot v = \mathbb{R}^2 \setminus \{x\}$, the set of all points on the plane other than x. In other words, a nontrivial "affine circle" centered at a point will just be the complement of that point.

5. Outlook

In mathematics, we are always trying to find a deeper understanding of interesting things. Of course, this prompts two important questions: how do we get a deeper understanding, and what is interesting?

Because the underlying geometry of a model (G, H) is determined by Lie-theoretic properties of G and H, so that one might (somewhat hyperbolically) describe geometry as "Lie theory with extra steps", we can guess that getting a better understanding of the Lie theory behind G and H will lead to a deeper understanding of the geometry. It is, moreover, certainly not unreasonable to guess that Lie-theoretically interesting choices of G and H will lead to interesting geometries.

In what follows, we will be looking at a particularly rich class of geometries called *parabolic geometries*. These will come from models whose model group is semisimple and whose isotropy is parabolic; we will review these terms and the relevant background over the next few lectures.

References

[1] Sharpe, R. W.: *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program.* Graduate Texts in Mathematics, vol. 166, Springer-Verlag, New York (1997)